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PHYSICAL MODEL OF A 6-CONSTANT  
ANISOTROPIC ELASTIC MATERIAL

by

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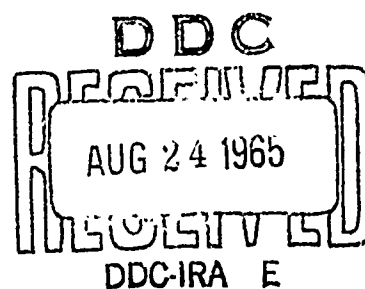
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## ABSTRACT

The physical model of a multi-constant elastic material [1]<sup>1</sup>, with consideration of couple-stresses, is studied critically to determine whether it more reasonably represents a 3-constant isotropic material or a 6-constant centro-symmetric cubic crystalline material. A description of a critical shear experiment on the model is presented along with experimental results. A parameter  $\alpha$  which is definitively related to the question is defined and its value determined on the basis of statical deformation experiments. The conclusion is that the model behaves as a cubic crystal showing couple-stress effects.

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<sup>1</sup> Numbers in brackets refer to references at end of report.

## INTRODUCTION

In a previous paper [1], a physical model of a so-called 3-constant isotropic elastic material was used to illustrate the phenomenon of couple-stresses. The model and its analysis were predicated on the assumption that couple-stresses exist and that the model represents an isotropic material, at least to some approximation. An obvious question provoked by the structured form of the model is whether it does not more accurately represent a cubic elastic material. As previously pointed out [1], Professor R. D. Mindlin strongly urged consideration of such a possibility. Consequently the present paper is intended to provide an analysis of the model from such a viewpoint. At first it is considered worthwhile to briefly review the classical theory of the constitutive equations of anisotropic materials and its extension to include considerations of couple-stresses.

## CONSTITUTIVE EQUATIONS OF GENERAL ANISOTROPIC SOLIDS

The classical theory of the deformation of crystals, without consideration of couple-stresses, has been extensively studied in the past [2, 3, 4]. The relationship of the concept of isotropy to that of a crystalline structure is a peculiar one. The concept of an isotropic body is not contained in that of the crystalline group and can only be

defined as a body resulting from a random orientation of crystals which produces a statistical equivalent to a solid continuum specified by two independent elastic constants. It is clear that if one contemplates the perfectly crystalline structure he diverges from the concept of isotropy. The group of crystals extends from the simplest, which is the centro-symmetric cubic, to the most complex, which is characterized by 21 elastic constants [4]. The defining device for crystalline structure is found in the mathematical transformations of rotation, reflection, and inversion [4]. In the classical theory, the constitutive equations relating the strain tensor and the stress tensor contain 81 constants which, because of symmetries and energy considerations, reduce to 21 constants in the case of the most complex crystal [2]. For the simplest crystal, the centro-symmetric cubical one, the independent constants reduce to 3 in number. As is well-known, there are two independent constants in the case of isotropic materials.

As long as couple-stresses are considered to be non-existent, the old classical theory of crystals is probably satisfactory for most purposes. However, once couple-stresses are postulated, as the present authors consider that they should be, the situation becomes more complicated. Kröner assumes the burden of this complexity in his attempt to explain deformation phenomena. He uses the idea of the motion of dislocations associated with the assumed existence of

couple-stresses to explain the phenomenon of plasticity. Furthermore, he also speculates that the deformation of the elastic framework of perfect crystals may introduce the need for couple-stresses in order to maintain equilibrium [5]. The view of the present authors, however, is that a student of continuum theory is not constrained to espouse any particular theory of the mechanical structure of matter but may at once resort to the generalization of the concept of continua by postulating the existence of couple-stresses. From this point of view the onus is on anyone who denies the existence of couple-stresses to demonstrate that they do not exist in any particular deformed body.

In any event the present paper is concerned with the proper definition of a cubic elastic material, in the presence of couple-stresses and its analysis. Furthermore, having determined the explicit form of the constitutive equations for such material the deformation of the physical model [1] will be studied to determine whether it approximates a cubic crystal in its action.

#### DERIVATION OF THE CONSTITUTIVE EQUATIONS FOR A 6-CONSTANT CUBIC CRYSTAL

For the purpose of clarity, a few words about the definition of stress and strain will now be given. Because the existence of couple-stresses is postulated in the theory, it

will be necessary to use symbols for the symmetric part and the anti-symmetric part of the stress tensor. Furthermore, because of the peculiar nature of couple-stresses, the couple-stress tensor will be divided into the deviatoric and the spherical parts [6].

The usual total resultant shearing and normal stresses will be designated by  $\tau_{ij}$  and written in terms of the symmetric and anti-symmetric parts as follows:

$$\tau_{ij} = \tau_{ij}^s + \tau_{ij}^a \quad (1)$$

where  $\tau_{ij}^s$  is the symmetric part and  $\tau_{ij}^a$  is the anti-symmetric part.

Furthermore, the couple-stresses will be designated by  $q_{ij}$  and written in terms of the deviatoric part and the spherical part as follows:

$$q_{ij} = q_{ij}^d + \frac{1}{3}(q_{kk})\delta_{ij} \quad (2)$$

where  $q_{ij}^d$  is the deviatoric part and the spherical part is represented by the terms in  $q_{kk}$ . The repeated index means summation over the 3-dimensional indices 1, 2, 3, as usual, and  $\delta_{ij}$  is the Kronecker delta.

Now  $\tau_{ij} \neq \tau_{ji}$  in general.

It may be observed that

$$\tau_{ij}^s = \frac{1}{2}(\tau_{ij} + \tau_{ji})$$

and

$$\tau_{ij}^a = \frac{1}{2}(\tau_{ij} - \tau_{ji})$$

(3)

The lineal and shearing strains, having the same meaning as in classical elasticity [7], will be designated by  $\epsilon_{ij}$  and are written in terms of the displacements  $u_i$  as follows:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (4)$$

where the comma means differentiation. The deformation corresponding to couple-stresses will be designated by  $K_{ij}$  and written as gradients of rotation as follows:

$$K_{ij} = e_{lmj} \omega_{m\ell,i} \quad (5)$$

where

$$\begin{aligned} e_{lmj} &= 0, \text{ when any two of the indices are equal} \\ &= +1, \text{ when } \ell, m, j \text{ is an even permutation of the} \\ &\quad \text{numbers } 1, 2, 3 \\ &= -1, \text{ when } \ell, m, j \text{ is an odd permutation of the} \\ &\quad \text{numbers } 1, 2, 3 \end{aligned}$$

$$\omega_{m\ell} = \frac{1}{2}(u_{m,\ell} - u_{\ell,m})$$

or as some investigators do we may write

$$K_{ij} = \omega_{j,i} \quad (6)$$

where in rectangular Cartesian coordinates the components of rotation are as follows:

$$\begin{aligned} \omega_1 &= \frac{1}{2} \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \\ \omega_2 &= \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \\ \omega_3 &= \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \end{aligned} \quad (7)$$

For a centro-symmetric general anisotropic material the energy function  $W$  may be written for an initially unstrained elastic body as follows:

$$W(\epsilon_{ij}, K_{ij}) = \frac{1}{2} S_{kl}^{ij} \epsilon_{ij} \epsilon^{kl} + \frac{1}{2} T_{lm}^{ij} K_{ij} K^{\ell m} \quad (8)$$

where  $S_{kl}^{ij}$  and  $T_{lm}^{ij}$  are elastic constants corresponding to the force-stresses and the couple-stresses respectively [6].

From the energy function expressions for the symmetric part of the stress components  $\tau_{ij}^s$  and the deviatoric part of the couple-stresses  $q_{ij}^d$  may be written as follows:

$$\tau_{ij}^s = \frac{\partial W}{\partial \epsilon_{ij}} = S_{ij}^{kl} \epsilon_{kl}$$

and

$$q_{ij}^d = \frac{\partial W}{\partial K_{ij}} = T_{ij}^{\ell m} K_{\ell m} \quad (9)$$



In a rectangular Cartesian system of coordinates:

$$\epsilon_{ij} = \epsilon^{ij}$$

and (10)

$$K_{ij} = K^{ij}.$$

Also the usual symmetry conditions can be proved [3]:

$$S_{kl}^{ij} = S_{ij}^{kl}$$

and (11)

$$T_{lm}^{ij} = T_{ij}^{lm}$$

The 4th order tensors  $S_{kl}^{ij}$  and  $T_{kl}^{ij}$  in indices which run over the values 1,2,3 for 3-space have apparently 81 components. However, as is well-known symmetry and thermodynamic conditions reduce the number radically. In fact, it is readily shown that the 81 values for  $S_{kl}^{ij}$  reduce to 21 by the symmetry relations [7] and the further symmetry relations:

$$S_{kl}^{ij} = S_{lk}^{ij}$$

and (12)

$$S_{kl}^{ij} = S_{kl}^{ji}$$

which result from the symmetry relations:

$$\tau_{ij}^s = \tau_{ji}^s$$

and (13)

$$\epsilon_{ij} = \epsilon_{ji}$$

The further reduction of the tensor  $S_{kl}^{ij}$  to represent the case of the centro-symmetric cubic crystal is given in standard texts on crystallography [4]. Then the three independent nonzero components turn out to be:

$$\begin{aligned} s_{11}^{11} &= s_{22}^{22} = s_{33}^{33} \\ s_{23}^{23} &= s_{13}^{13} = s_{12}^{12} \\ s_{22}^{11} &= s_{33}^{11} = s_{33}^{22} \end{aligned} \tag{14}$$

Using contracted notation for simplicity of writing the explicit form of that portion of the constitutive equations which derive only from forces and not from couples, may be written:

$$\begin{aligned} \tau_{11}^s &= s_{11} \epsilon_{11} + s_{12} \epsilon_{22} + s_{12} \epsilon_{33} \\ \tau_{22}^s &= s_{12} \epsilon_{11} + s_{11} \epsilon_{22} + s_{12} \epsilon_{33} \\ \tau_{33}^s &= s_{12} \epsilon_{11} + s_{12} \epsilon_{22} + s_{11} \epsilon_{33} \\ \tau_{23}^s &= s_{44} \epsilon_{23} \\ \tau_{13}^s &= s_{44} \epsilon_{13} \\ \tau_{12}^s &= s_{44} \epsilon_{12} \end{aligned} \tag{15}$$

Now to derive that portion of the constitutive equations which derive only from consideration of couples at a point, it is necessary to take into account the asymmetry of  $K_{ij}$

and  $q_{ij}^d$  as well as the geometrical fact that:

$$K_{ii} = 0 \quad (16)$$

which implies that

$$q_{ii}^d = 0 \quad (17)$$

[again repeated index implies summation].

It is then clear that the reduction of the 4th order tensor  $T_{k\ell}^{ij}$  is formally somewhat different from the reduction of  $S_{k\ell}^{ij}$ . From the conditions (16) and (17) on  $K_{ii}$  and  $q_{ii}^d$  respectively the following nine conditions are imposed on  $T_{k\ell}^{ij}$ :

$$T_{ij}^{kk} = 0 \quad (18)$$

where  $i, j$  take on the values  $1, 2, 3$  and the repeated index  $k$  implies summation.

From the methodical application of the transformations of rotation, reflection, and inversion, the explicit form of that portion of the constitutive equations which derives from consideration of couples at a point may be written as follows:

$$\begin{aligned}
q_{11}^d &= \frac{3A}{2} K_{11} \\
q_{22}^d &= \frac{3A}{2} K_{22} \\
q_{33}^d &= \frac{3A}{2} K_{33} \\
q_{23}^d &= CK_{23} + BK_{32} \\
q_{32}^d &= CK_{32} + BK_{23} \\
q_{13}^d &= CK_{13} + BK_{31} \\
q_{31}^d &= CK_{31} + BK_{13} \\
q_{12}^d &= CK_{12} + BK_{21} \\
q_{21}^d &= CK_{21} + BK_{12}
\end{aligned} \tag{19}$$

where A, B, C are independent constants.

If the material were isotropic, the following conditions apply:

$$S_{11} - S_{12} = S_{44}$$

and (20)

$$\frac{3A}{2} = B + C$$

For a centro-symmetric cubic crystal the six independent elastic constants are then  $S_{11}$  ,  $S_{12}$  ,  $S_{44}$  , A, B, and C .

As a first step beyond the theory of classical crystallography which requires only the constants  $S_{11}$  ,  $S_{12}$  ,  $S_{44}$  , it seems appropriate to place  $B$  equal to zero. In this event, the number of nonzero independent constants is apparently five, which are the three corresponding to classical elasticity and the two introduced because of consideration of couples at a point. These latter two constants,  $A$  and  $C$  , correspond to twisting couples and flexure couples respectively.

#### PHYSICAL MODEL OF A 6-CONSTANT CUBIC ELASTIC MATERIAL

As observed in the introduction to the present paper, one of its principal objectives is to examine the question of whether a physical model used previously [1] represents a 3-constant isotropic material really or more appropriately a 6-constant cubic elastic material. A crucial test consists of a determination of the value of a parameter  $\alpha$  given as follows [9]:

$$\alpha = \frac{S_{44}}{S_{11} - S_{12}} \quad (21)$$

If  $\alpha$  equals unity, the first condition (20) is satisfied and the material is presumed to be isotropic. On the other hand if  $\alpha$  is not equal to unity, with due consideration of margins of experimental error, the material will be presumed to be cubic crystalline.

As it turns out, the tension experiments previously devised and conducted [1] properly determine the values of the constants  $S_{11}$  ,  $S_{12}$  whether the body is isotropic or cubic. Therefore only a single additional experiment, for the determination of the shear modulus  $S_{44}$  , is required. The displacement functions and the shear experiment will now be described. For convenient reference some details of the physical model are shown in Fig. 1.

#### EXPERIMENT TO DETERMINE ELASTIC SHEAR CONSTANT $S_{44}$

In order to evaluate the shear constant  $S_{44}$  it is sufficient to assume that a rectangular prism is loaded around its lateral faces with uniformly distributed shearing stresses of constant magnitude, say  $\tau$  as shown in Fig. 2. The displacements caused by such a loading may be obtained theoretically by solution of the corresponding plane strain problem of an infinitely long rectangular prism. It is assumed satisfactory for experimental purposes, to an acceptable degree of approximation, to represent such a theoretical situation by a slice of the prism.

In order to obtain theoretically determined displacement functions it is necessary to solve the six equations of equilibrium [1, 8], three on force and three on moments. Obviously for the present assumed shape and simple loading, these equations reduce to the following two:

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} = 0$$

and (22)

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} = 0$$

The only compatibility condition which is not identically satisfied is as follows:

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \quad (23)$$

The solutions of this simple boundary value problem, in terms of stresses, can be seen to be:

$$\begin{aligned} \tau_{11} &= 0 \\ \tau_{22} &= 0 \\ \tau_{12} &= \tau_{21} = \tau = \text{constant} \end{aligned} \quad (24)$$

Now, as usual, an integration of the strain-displacement equations will provide the necessary equations for the determination of the shear modulus  $S_{44}$ . The equations are:

$$\begin{aligned}
\frac{\partial u_1}{\partial x_1} &= 0 \\
\frac{\partial u_2}{\partial x_2} &= 0 \\
\text{and } \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) &= \frac{\tau}{S_{44}}
\end{aligned} \tag{25}$$

Consequently one obtains for the displacement functions  $u_1$  and  $u_2$  the following:

$$\begin{aligned}
u_1 &= \frac{\tau}{S_{44}} x_2 \\
\text{and } u_2 &= \frac{\tau}{S_{44}} x_1
\end{aligned} \tag{26}$$

The displacement  $(u_1, u_2)$  can be measured at any point  $(x_1, x_2)$  with the model loaded by constant shearing stress  $\tau$ . Consequently,  $S_{44}$  can then be evaluated.

A model consisting of metallic cubes connected by either flat steel strips or round steel rods was designed and fabricated. It is of the same type as the one previously described in the literature [1]. For the present investigation, two types of connector pieces, flat and round, were used. The reason for making model slices with the round rods is that it is planned to build up later large prisms which are essentially 3-dimensional models for study purposes. Such models will



enable one to measure the final elastic constant, the twist modulus  $A$ , which is still undetermined.

As one may recall from the previous paper [1], the flat steel connector pieces had cross-sectional dimensions 0.5" by 0.016". The round steel rods used in the second model are 0.095 inches in diameter and 2 inches between cubes.

In order to perform the experiments each model was mounted horizontally so that it rested on several steel ball bearings in order to maintain itself in the horizontal plane when it was loaded and at the same time minimize any frictional forces which might oppose the free deformation.

In order to deform the model as required, loads were applied through special fittings such that they were uniformly distributed through the thickness and along the sides. The experimental apparatus and model are shown in Fig. 3. The elongation and contraction of the two diagonals were measured using special displacement gages [1]. Elongation of the one diagonal and simultaneous contraction of the other were practically equal in magnitude as expected. However, on account of small differences the two measured lengths were averaged in order to calculate the change in angle  $\gamma$  at any corner of the model. Measured values are shown in Fig. 4. For infinitesimal deformation theory, the change in angle is given by the expression:

$$\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \quad (27)$$

Hence knowing the value of the change of angle, the value of the constant  $S_{44}$  can be readily calculated from Eq. (26).

An additional experiment was conducted on the model with round rod connectors in order to determine the values of  $S_{11}$  and  $S_{12}$ .

All of the elastic constants so far determined by means of experiments are shown in Table I.

#### DISCUSSION AND CONCLUSIONS

Since the pertinent elastic constants have been determined, the decisive constant  $\alpha$  needed to determine whether the model may be considered isotropic or cubic crystalline can be calculated. Its value for the first model is  $19 \times 10^{-3}$  and for the second model  $15.8 \times 10^{-3}$ .

It is clear that since the values of  $\alpha$  are so radically different from unity, the models cannot be considered as representing an isotropic material. So far as the evidence goes they may be considered as models of the somewhat more complex cubic crystalline matter.

It should be recalled that the value of the constant  $S_{12}$  for the present type of model is approximately zero.

The last constant associated with a cubic crystalline material exhibiting the phenomenon of couple-stresses is the twist constant which has been designated  $A$  in the present paper. For the purpose of determining the value of this constant it will be necessary to fabricate a 3-dimensional model built up out of the cubelike substructures. Such a model, of sufficiently large proportions, is now being constructed and probably will be subjected to an experiment in the near future.

In addition to experiments for the determination of the twist modulus  $A$ , vibration and elastic wave transmission experiments are planned.

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Table I. Elastic Constants

Model	Tensile Modulus $S_{11}$ h lbs./in.	Shear Modulus $S_{44}$ h lbs./in.	Flexure Modulus Ch lbs./in.
With steel strips as interconnection	$.182 \times 10^6$	3572	$.295 \times 10^5$
With steel rods as interconnection	$.132 \times 10^6$	2080	$.777 \times 10^5$

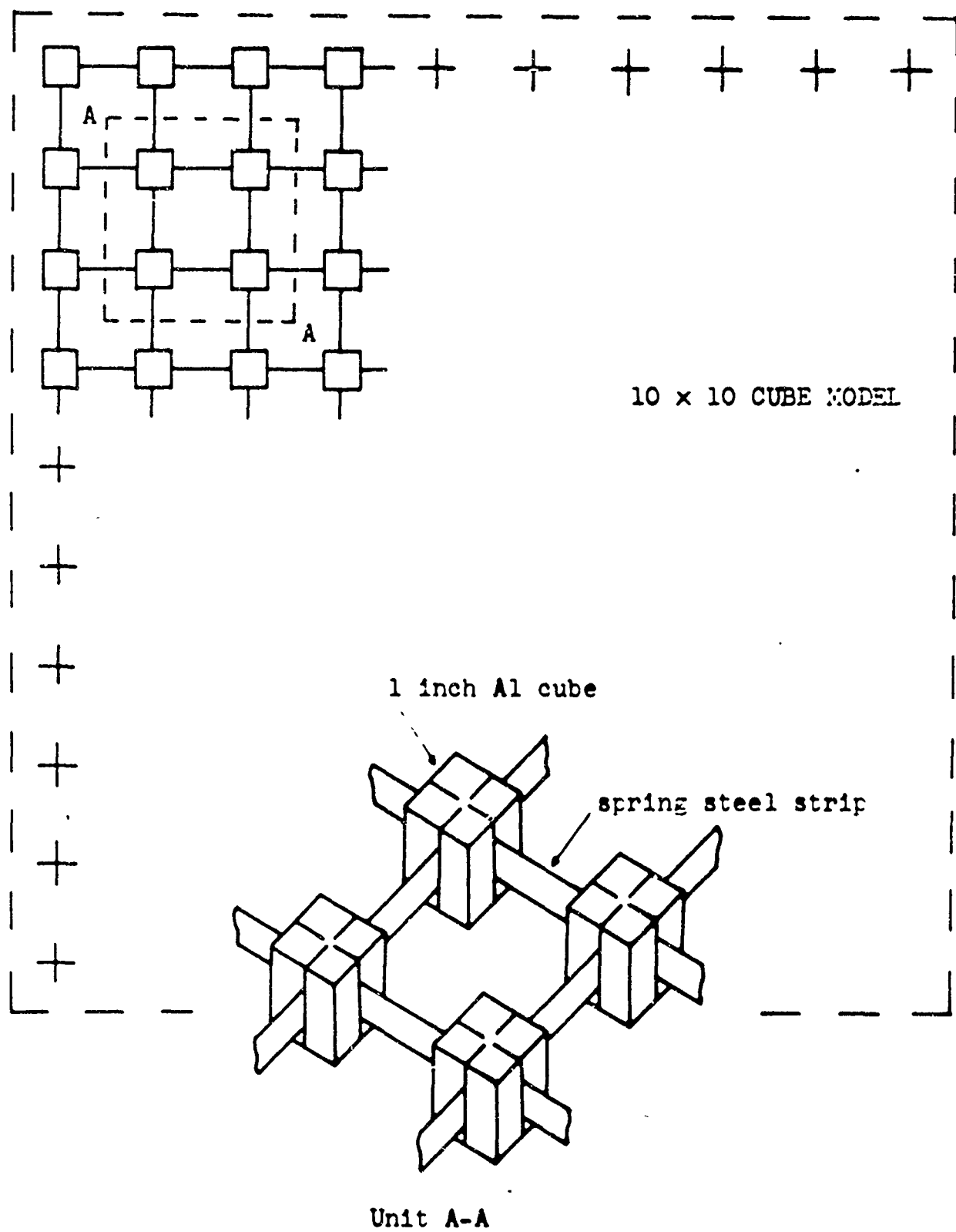


Fig. 1. Some Details of Physical Model

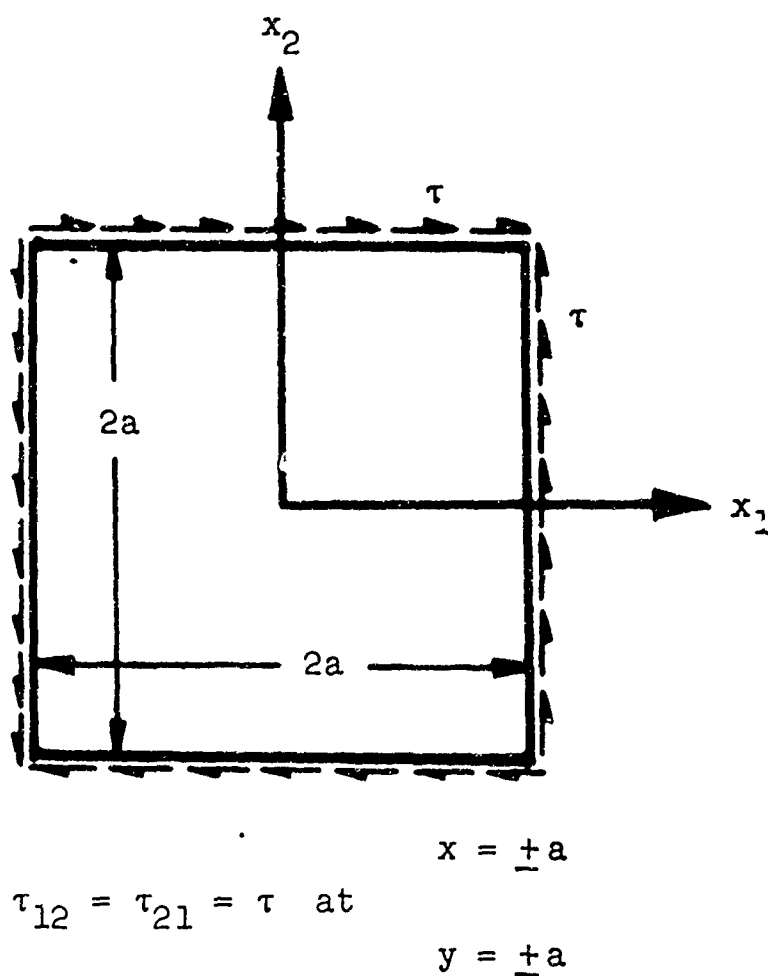


Fig. 2. Section of Model Showing  
Type of Loading

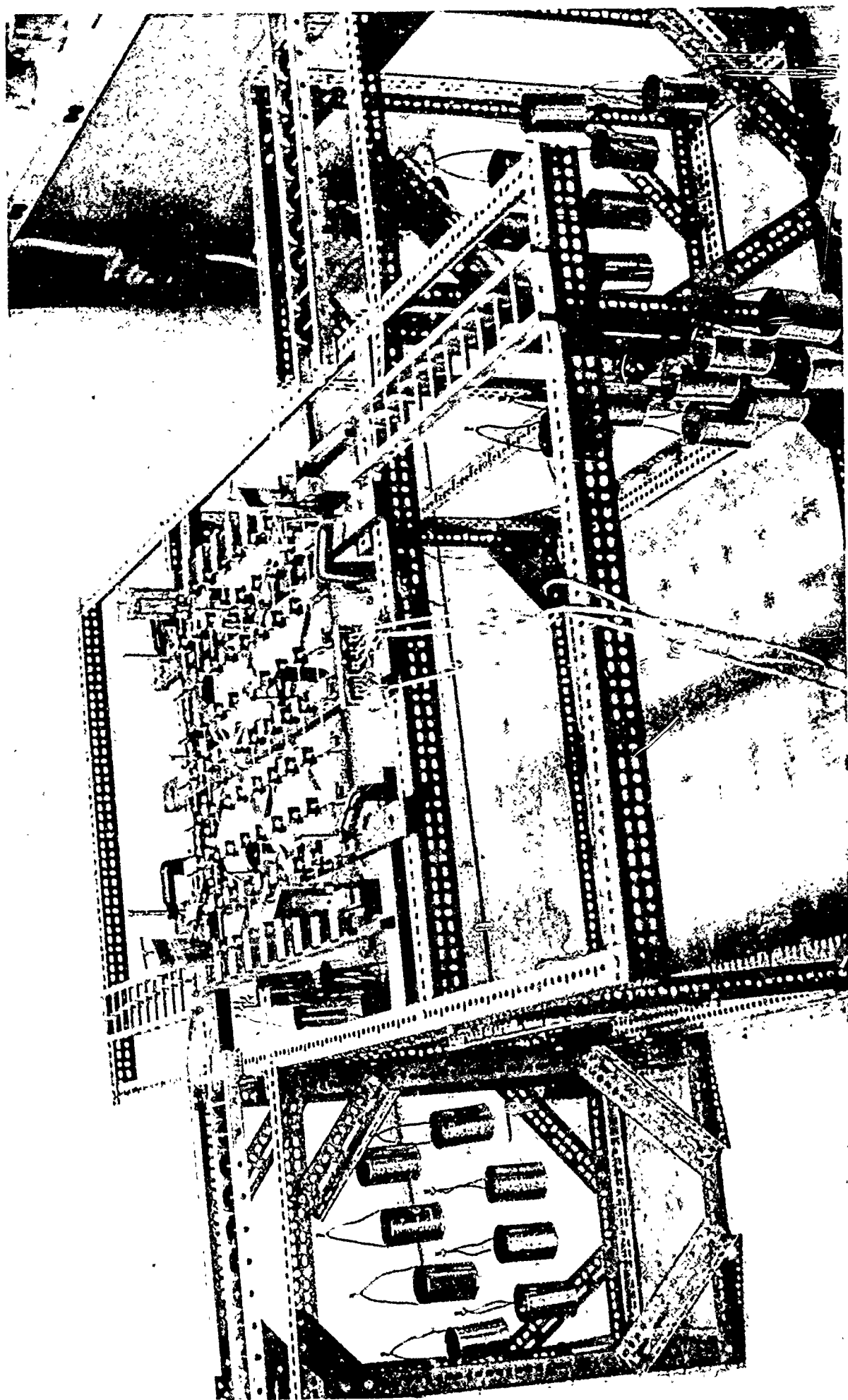


Fig. 3. Photograph of the Experimental Apparatus and Model



